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A NEW SOLUTION OF KIRCHHOFF'S EQUATIONS IN THE CASE OF A LINEAR INVARIANT RELATION†

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The problem of integrating Kirchhoff's differential equations [1] when they allow of a linear invariant relation with respect to the main variables – the components of the angular momentum of a gyrostat and the unit vector of the axis of symmetry of the force field, is considered. The initial system of equations is reduced to a second-order system using first integrals of the equations. Under certain conditions, imposed on the parameters characterizing the geometry of the gyrostat masses and the potential and gyroscopic forces, the integrating factor of the reduced equations is obtained. The solution of Kirchhoff's equations obtained contains four arbitrary constants and is determined for more general assumptions compared with existing solutions [2–4]. © 2006 Elsevier Ltd. All rights reserved.

In the dynamics of a rigid body with a fixed point, it is of considerable interest to investigate not only the classical problem of the motion of a heavy rigid body [5, 6], but also various generalizations of it, particularly the problem of the motion of a gyrostat acted upon by potential and gyroscopic forces and the problem of the motion of a heavy rigid body in an ideal incompressible fluid [1–4, 8, 9]. This is due to the fact that these two problems are mathematically equivalent, since they are described by Kirchhoff-class differential equations [1, 7]. In view of this fact, the previously established cases of the integrability of the equations of motion of a solid in a fluid (see the reviews in [5, 6] and also [3, 4, 9, 10]) can be interpreted as solutions of the equations of motion of a gyrostat acted upon by potential and gyroscopic forces [7]. Since the inverse assertion also holds, any new solution of the equations of motion of a gyrostat [7] is also a new solution of Kirchhoff's equations.

In the general case, the non-integrability of the Euler–Poisson equations was proved in [11] and the non-integrability of the Kirchhoff–Poisson equations was proved in [8]. The problem of constructing new particular solutions for these equations is therefore a pressing problem [6].

1. FORMULATION OF THE PROBLEM

Consider the differential equations of the motion of a gyrostat with a fixed point, in a force field, which is the superposition of Newtonian, electric and magnetic fields, in the formulation proposed previously in [7]

$$\dot{\mathbf{x}} = (\mathbf{x}+\boldsymbol{\lambda})\times a\mathbf{x} + a\mathbf{x}\times B\mathbf{v} + \mathbf{s}\times\mathbf{v} + \mathbf{v}\times C\mathbf{v} \tag{1.1}$$

$$\dot{\mathbf{v}} = \mathbf{v}\times a\mathbf{x} \tag{1.2}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is the angular momentum of the gyrostat $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ is the unit vector of the axis of symmetry of the force fields, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is the gyrostatic moment, characterizing the motion of the carrier bodies, $\mathbf{s} = (s_1, s_2, s_3)$ is a vector codirectional with the vector of the generalized centre of mass, $a = (a_{ij})$ is the gyration tensor, constructed at a fixed point, $B = (B_{ij})$ is a constant symmetrical

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third-order matrix, defining the gyroscopic forces, and $C = (C_{ij})$ is a constant symmetrical third-order matrix, characterizing the potential forces.

Equations (1.1) and (1.2) have first integrals

$$\mathbf{x}\cdot a\mathbf{x} - 2(\mathbf{s}\cdot\mathbf{v}) + (C\mathbf{v}\cdot\mathbf{v}) = 2E, \quad \mathbf{v}\cdot(\mathbf{x}+\boldsymbol{\lambda}) - (B\mathbf{v}\cdot\mathbf{v})/2 = k, \quad \mathbf{v}\cdot\mathbf{v} = 1 \tag{1.3}$$

Here E and k are arbitrary constants.

We will formulate the problem of determining the conditions for the existence in system (1.1), (1, 2) of a single invariant relation

$$x_1 - (g_0 + g_1\nu_1 + g_2\nu_2 + g_3\nu_3) = 0 \tag{1.4}$$

where g_i ($i = 0, 1, 2, 3$) are constants to be determined.

For the problem of the motion of a body in a fluid the conditions for the existence of invariant relation (1.4) were obtained in [4], but the integration of the Kirchhoff–Poisson equations was only carried out in special cases [2, 3], where the version $\boldsymbol{\lambda} = \mathbf{0}$, $\mathbf{s} = \mathbf{0}$ was investigated in [2] and the version $\boldsymbol{\lambda} = \mathbf{0}$ was investigated in [3]. For the classical problem of the motion of a heavy rigid body, the equations of which follow from system (1.1), (1.2) with $\boldsymbol{\lambda} = \mathbf{0}$, $B = 0$ and $C = 0$, the analogue of relation (1.4), namely, $x_1 = 0$, was investigated by Hess in [12]. A geometrical interpretation of Hess' solution was given by Kovalev [13]. Sretenskii extended this solution [14] to the case when $\boldsymbol{\lambda} \neq \mathbf{0}$, $B = 0$, $C = 0$.

We will differentiate relation (1.4) by virtue of the scalar equations which follow from system (1.1), (1.2), and we will require that the equality obtained, after substituting relation (1.4) into it, should be an identity for any values of the variables $x_2, x_3, \nu_1, \nu_2, \nu_3$. We then obtain the following conditions, imposed on the parameters of the problem and the parameter g_1 ($i = 0, 1, 2, 3$)

$$\begin{gathered}
a_{12} = a_{23} = 0, \quad a_{22} = a_{33}, \quad \lambda_2 = 0, \quad a_{13}g_0 - a_{22}\lambda_3 = 0, \quad g_2 = B_{12} \\
a_{13}g_1 - a_{22}g_3 + a_{22}B_{13} = 0, \quad a_{13}g_2 + a_{22}B_{23} = 0 \\
a_{13}g_3 + a_{22}g_1 + a_{22}B_{33} = 0, \quad a_{22}g_1 - a_{13}g_3 + a_{22}B_{22} = 0 \\
s_2 = g_0(a_{13}B_{23} + a_{11}g_2), \quad s_3 = g_0(a_{11}g_3 - a_{13}g_1 - a_{13}B_{22}) \\
C_{12} + g_1(a_{13}B_{23} + a_{11}g_2) = 0, \quad C_{13} + g_1(a_{11}g_3 - a_{13}g_1 - a_{13}B_{22}) = 0 \\
C_{23} + g_2(a_{11}g_3 - a_{13}g_1 - a_{13}B_{22}) = 0, \quad C_{23} + g_3(a_{13}B_{23} + a_{11}g_2) = 0 \\
C_{22} - C_{33} = a_{11}(g_3^2 - g_2^2) - a_{13}(g_1g_3 + g_3B_{22} + g_2B_{23})
\end{gathered} \tag{1.5}$$

We will consider the version when $a_{13} = 0$. Without loss of generality we can assume that $s_3 = 0$. Then, putting $a_{ii} = a_i$ ($i = 1, 2, 3$), we have from system (1.5)

$$\begin{gathered}
a_{ij} = 0 \quad (i \neq j), \quad a_3 = a_2, \quad \lambda_2 = \lambda_3 = 0, \quad B_{23} = B_{13} = 0, \quad B_{33} = B_{22} \\
C_{12} = a_1B_{12}B_{22}, \quad C_{23} = C_{13} = 0, \quad C_{33} - C_{22} = a_1B_{12}^2, \quad s_3 = 0 \\
g_0 = s_2/(a_1B_{12}), \quad g_1 = -B_{22}, \quad g_2 = B_{12}, \quad g_3 = 0
\end{gathered} \tag{1.6}$$

It follows from the conditions $a_{ij} = 0$, $a_3 = a_2$ that the first coordinate axis, with respect to which the linear invariant relation (1.4) is specified, is orthogonal to the circular section of the gyration ellipsoid. The other equalities of (1.6) show that the vector of the gyrostatic moment is directed along the same axis, while the vector s, in general, does not belong to it.

On the basis of conditions (1.6), relation (1.4) can be written as follows:

$$x_1 = g_0 - B_{22}\nu_1 + B_{12}\nu_2 \tag{1.7}$$

Equations (1.1) and (1.2), with conditions (1.6) and (1.7), take the form

$$\begin{aligned}
\dot{x}_2 &= x_3(\alpha_0 + \alpha_1 \nu_1 + \alpha_2 \nu_2) - \alpha_3 \nu_3 - \alpha_{13} \nu_1 \nu_3 \\
\dot{x}_3 &= -x_2(\alpha_0 + \alpha_1 \nu_1 + \alpha_2 \nu_2) + \alpha_3 \nu_2 + \alpha_{13} \nu_1 \nu_2 \\
\dot{\nu}_1 &= a_2(x_3 \nu_2 - x_2 \nu_3) \\
\dot{\nu}_2 &= -a_2 x_3 \nu_1 + a_1 g_0 \nu_3 - a_1 B_{22} \nu_1 \nu_3 + a_1 B_{12} \nu_2 \nu_3 \\
\dot{\nu}_3 &= a_2 x_2 \nu_1 - a_1 g_0 \nu_2 + a_1 B_{22} \nu_1 \nu_3 - a_1 B_{12} \nu_2^2
\end{aligned} \tag{1.8}$$

where

$$\alpha_0 = g_0(a_1 - a_2) - a_2 \lambda_1, \quad \alpha_1 = a_2 B_{11} - (a_1 - a_2) B_{22}$$

$$\alpha_2 = a_1 B_{12}, \quad \alpha_3 = s_1 + a_{22} g_0 B_{22}, \quad \alpha_{13} = C_{22} - C_{11} + a_1 B_{12}^2 - a_1 B_{22}^2$$

We will take relations (1.6) and (1.7) into account in the integrals (1.3). We obtain

$$\begin{aligned}
& a_2(x_2^2 + x_3^2) - 2\alpha_3 \nu_1 - \alpha_{13} \nu_1^2 = 2E_1, \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1 \\
& x_2 \nu_2 + x_3 \nu_3 + (g_0 + \lambda_1)\nu_1^2 - (B_{11} + B_{22})\nu_1^2/2 = k_1
\end{aligned} \tag{1.9}$$

Here

$$E_1 = E - (a_1 g_0^2 + C_{33})/2, \quad k_1 = k + B_{22}/2$$

(E_1 and k_1 are new arbitrary constants).

Equations (1.8) have three first integrals (1.9), and hence their integration reduces to the integration of a second-order system. We will find the integrating factor of this system using Jacobi's integrating factor theory [15]. This approach is usually employed when we know the additional first integral of system (1.1), (1.2), since the Jacobi integrating factor in it is equal to unity. The following condition holds for system (1.8)

$$\sum_{i=1}^{5} \frac{\partial Y_i}{\partial y_i} \neq 0$$

where

$$y_1 = x_2, \quad y_2 = x_3, \quad y_3 = \nu_1, \quad y_4 = \nu_2, \quad y_5 = \nu_3$$

and Y_i are the right-hand sides of system (1.8). Hence, the Jacobi integrating factor is a function of the variables y_i ($i = 1, 2, \dots, 5$). In view of that the use of the general theory [15] is considerably, and hence in this paper we will integrate system (1.8) by finding the integrating factor of the second-order reduced system.

From the first and third relations of system (1.9) we have

$$x_2 = \frac{a_2 \nu_2 \varphi(\nu_1) + \nu_3 \sqrt{\Delta(\nu_1)}}{a_2(1 - \nu_1^2)}, \quad x_3 = \frac{a_2 \nu_3 \varphi(\nu_1) - \nu_2 \sqrt{\Delta(\nu_1)}}{a_2(1 - \nu_1^2)} \tag{1.10}$$

where

$$\begin{aligned}
\varphi(\nu_1) &= k_1 - (g_0 + \lambda_1)\nu_1 + (B_{11} + B_{22})\nu_1^2/2 \\
\Delta(\nu_1) &= d_0 + d_1 \nu_1 + d_2 \nu_1^2 + d_3 \nu_1^3 + d_4 \nu_1^4
\end{aligned} \tag{1.11}$$

Here

$$\begin{aligned} d_0 &= a_2(2E_1 - a_2k_1), \quad d_1 = 2a_2[\alpha_3 + a_2(g_0 + \lambda_1)k_1] \\ d_2 &= a_2[\alpha_{13} - 2E_1 - a_2(g_0 + \lambda_1)^2 - a_2k_1(B_{11} + B_{22})] \\ d_3 &= a_2[a_2(g_0 + \lambda_1)(B_{11} + B_{22}) - 2\alpha_3] \\ d_4 &= -a_2[\alpha_{13} + a_2(B_{11} + B_{22})^2/4] \end{aligned} \tag{1.12}$$

We introduce expressions (1.7) and (1.10) into the last three equations of system (1.8). We obtain

$$\dot{\nu}_1 = -\sqrt{\Delta(\nu_1)}, \quad \dot{\nu}_2 = \frac{\nu_1\nu_2\sqrt{\Delta(\nu_1)} + \nu_3\psi(\nu_1)}{1 - \nu_1^2}, \quad \dot{\nu}_3 = \frac{\nu_1\nu_3\sqrt{\Delta(\nu_1)} - \nu_2\psi(\nu_1)}{1 - \nu_1^2} \tag{1.13}$$

Here

$$\begin{aligned} \psi(\nu_1) &= a_1B_{12}\nu_2(1 - \nu_1^2) + P(\nu_1), \quad P(\nu_1) = p_0 + p_1\nu_1 + p_2\nu_1^2 + p_3\nu_1^3 \\ p_0 &= a_1g_0, \quad p_1 = -a_2k_1 - a_1B_{22}, \quad p_2 = a_2\lambda_1 + g_0(a_2 - a_1) \\ p_3 &= [2a_1B_{22} - a_2(B_{11} + B_{22})]/2 \end{aligned} \tag{1.14}$$

Hence, the integration of Eqs (1.1) and (1.2) over invariant relation (1.4) reduces to integrating the third-order system (1.13).

2. INTEGRATION OF SYSTEM (1.13)

We will introduce the variables φ and θ instead of the variables ν_1, ν_2 and ν_3. By virtue of the geometrical integral $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$ we can write

$$\nu_1 = \cos\theta, \quad \nu_2 = \sin\theta\cos\varphi, \quad \nu_3 = \sin\theta\sin\varphi \tag{2.1}$$

We substitute expressions (2.1) into system (1.3). We obtain

$$d\theta/dt = \sqrt{\Delta(\cos\theta)}/\sin\theta \tag{2.2}$$

$$\sqrt{\Delta(\cos\theta)}\sin\theta d\varphi + [a_1B_{12}\sin^3\theta\cos\varphi + P(\cos\theta)]d\theta = 0 \tag{2.3}$$

Equation (2.2) defines the relation $\theta(t)$. In general, it follows from this that $\nu_1 = \cos\theta$ is an elliptic function of time. To determine the function $\varphi(\theta)$ from Eq. (2.3) we specify the integrating factor of this equation in the form

$$M(\varphi, \theta) = [\sqrt{\Delta(\cos\theta)}N(\varphi, \theta)]^{-1}, \quad N(\varphi, \theta) = \varphi_1(\theta)\sin\varphi + \varphi_2(\theta)\cos\varphi + \varphi_3(\theta) \tag{2.4}$$

where $\varphi_i(\theta)$ ($i = 1, 2, 3$) are functions to be determined.

If the integrating factor (2.4) is found, Eq. (2.3) can be written in the form

$$\frac{\partial V(\varphi, \theta)}{\partial \varphi}d\varphi + \frac{\partial V(\varphi, \theta)}{\partial \theta}d\theta = 0 \tag{2.5}$$

Here, by virtue of relations (2.3) and (2.4), we have put

$$\frac{\partial V(\varphi, \theta)}{\partial \varphi} = \frac{\sin\theta}{N(\varphi, \theta)}, \quad \frac{\partial V(\varphi, \theta)}{\partial \theta} = \frac{a_1B_{12}\sin^3\theta\cos\varphi + P(\cos\theta)}{\sqrt{\Delta(\cos\theta)}N(\varphi, \theta)} \tag{2.6}$$

Bearing these equations in mind, we can write the equality

$$\frac{\partial^2 V(\varphi, \theta)}{\partial\theta\partial\varphi} = \frac{\partial^2 V(\varphi, \theta)}{\partial\varphi\partial\theta}$$

We have

$$\sqrt{\Delta(\cos\theta)}(\varphi_3'(\theta)\sin\theta - \varphi_3(\theta)\cos\theta) = a_1 B_{12}\varphi_1(\theta)\sin^3\theta \tag{2.7}$$

$$\sqrt{\Delta(\cos\theta)}(\varphi_2'(\theta)\sin\theta - \varphi_2(\theta)\cos\theta) = \varphi_1(\theta)P(\cos\theta) \tag{2.8}$$

$$\sqrt{\Delta(\cos\theta)}(\varphi_1'(\theta)\sin\theta - \varphi_1(\theta)\cos\theta) = a_1 B_{12}\varphi_3(\theta)\sin^3\theta - \varphi_2(\theta)P(\cos\theta) \tag{2.9}$$

To find the solution of system (2.7)–(2.9), by analogy with [2, 3] we put $\varphi_1(\theta) = \sqrt{\Delta(\cos\theta)}$. Equation (2.7) can then be simply integrated

$$\varphi_3(\theta) = \Phi(\cos\theta)\sin\theta, \quad \Phi(\cos\theta) = x_0 - a_1 B_{12}\cos\theta \tag{2.10}$$

where x_0 is an arbitrary constant. The solution of Eq. (2.8) will be found in the class of polynomials in $\cos\theta$. It can be shown that, when $p_2 = 0$, it allows of the solution

$$\varphi_2(\theta) = -(p_1 + 2p_3) - p_0\cos\theta + p_3\cos^2\theta \tag{2.11}$$

Hence, by virtue of the notation (1.14) we obtain, in addition to conditions (1.6), the constraint of the parameters of problem (1.1), (1.2)

$$\lambda_1 = g_0(a_1 - a_2)/a_2 \tag{2.12}$$

We introduce the expressions $\varphi_1(\theta) = \sqrt{\Delta(\cos\theta)}$, (2.10) and (2.11) into Eq. (2.9) and require that the equality obtained should be an identity in θ. Then, in view of the expression for $\Delta(\nu_1)$ we obtain from relations (1.11)

$$\begin{aligned}
&d_1 + 2x_0 a_1 B_{12} + 2p_0(p_1 + 2p_3) = 0\\
&d_0 + d_2 - a_1^2 B_{12}^2 + p_0^2 + p_1(p_1 + 2p_3) = 0\\
&d_1 + 3d_3 - 4a_1 B_{12} x_0 - 2p_0 p_3 + 2p_0 p_1 = 0\\
&d_4 + a_1^2 B_{12}^2 + p_3^2 = 0, \quad d_3 - 2a_1 B_{12} x_0 - 2p_0 p_3 = 0
\end{aligned} \tag{2.13}$$

It can be shown on the basis of the notation (1.12), (1.14) and condition (2.12) that the system of equations (2.13) is dependent and can be reduced to only two equations

$$a_2(C_{33} - C_{11}) = a_1(a_1 B_{22} - a_2 B_{11})B_{22} + a_1^2 B_{12}^2 \tag{2.14}$$

$$x_0 = [a_1 g_0(a_2 B_{11} - a_1 B_{22}) - a_2 s_1]/(a_1 B_{12}) \tag{2.15}$$

Hence, if the parameters of problem (1.1), (1.2) satisfy conditions (2.12) and (2.14), the differential equation (2.3) allows of the integrating factor (2.4), where $\varphi_1(\theta) = \sqrt{\Delta(\cos\theta)}$, $\varphi_2(\theta)$ and $\varphi_3(\theta)$ are expressed by formulae (2.10) and (2.11), while the parameter x_0 is defined by relations (2.15). The constants E_1 and k_1, occurring in the expression for d_i from (1.12), remain arbitrary.

We return to system of equations (2.6). From the first equation of this system, in general, one can obtain three versions of the result of the integration. They depend on the values of the quantity

$$\mu_0 = (p_1 + 2p_3)^2 + d_0 - x_0^2$$

Here we assume that the inequality $\mu_0 > 0$ is satisfied, which can be achieved by the choice of the arbitrary constants E_1 and k_1. Then, by virtue of relations (1.11) and (1.12), it follows from Eq. (2.2) that values of the variable θ exist for which the quantity $\Delta(\cos\theta)$ is not negative, i.e. $\theta(t)$ is a real function of time. When $\mu_0 > 0$ we have

$$V(\varphi, \theta) = \frac{1}{\sqrt{\mu_0}}\ln\left|\frac{H_+(\varphi, \theta)}{H_-(\varphi, \theta)}\right| + F(\theta) \tag{2.16}$$

where

$$H_\pm(\varphi, \theta) = h(\theta) \pm \sqrt{\mu_0}\operatorname{tg}\frac{\varphi - \alpha(\theta)}{2}, \quad h(\theta) = \Phi(\cos\theta) + \sqrt{\mu_0 + \Phi^2(\cos\theta)}$$

$$\alpha(\theta) = \arccos\frac{\varphi_2(\cos\theta)}{\sin\theta\sqrt{\mu_0 + \Phi^2(\cos\theta)}}$$

while $F(\theta)$ is a function which is found by substituting expression (2.16) into the second equation of system (2.6)

$$F(\theta) = f(\nu_1(\theta)) = \int\frac{-a_1 B_{12}[(p_1 + 2p_3) + p_0\nu_1 - p_3\nu_1^2]}{[\mu_0 + \Phi^2(\nu_1)]\sqrt{\Delta(\nu_1)}}d\nu_1 \tag{2.17}$$

Hence, relation (2.5) takes the form $dV(\varphi, \theta) = 0$, i.e. Eq. (2.3) allows of the first integral $V(\varphi, \theta) = C$ when conditions (2.12), (2.14) and (2.15) are satisfied, where C is an arbitrary constant. By virtue of equalities (2.16) and (2.17) we can reduce it to the form

$$\varphi(\theta) = \alpha(\theta) + 2\operatorname{arctg}\left[\frac{h(\theta)}{\sqrt{\mu_0}}\operatorname{th}\frac{\sqrt{\mu_0}}{2}(C - f(\cos\theta))\right] \tag{2.18}$$

Consequently, in the solution constructed $\nu_1 = \cos\theta(t)$ is an elliptic function of time, while the relation $\varphi = \varphi(\theta)$ is given by Eq. (2.18). After substituting these functions into Eqs (1.7), (1.10) and (2.1), we obtain the dependence of all the variables of problem (1.1), (1.2) on the time t. The solution obtained depends on four arbitrary constants E, k, C and t_0.

In conclusion, we note that Chaplygin [2] constructed a solution of Eqs (2.2) and (2.3) for the conditions $g_0 = 0$, $\lambda_1 = 0$ and $s_1 = 0$, while Kharlamov [3] constructed a solution for the conditions $g_0 = 0$ and $\lambda_1 = 0$.

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